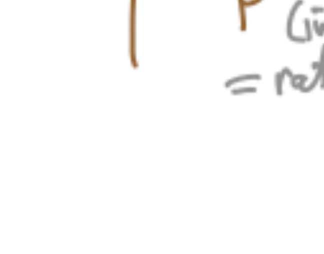
Our gradient vector ∇F for a function $F(x, y, z, \ldots)$

is $\nabla F = (F_x, F_y, F_z, \ldots)$

- a vector field (a vector at every point).
- points in the direction where F is increasing fastest.
- it's magnitude is the rate of change in that direction.
- ∇F is $\perp$ to the level set $F(x,y,z,\ldots) = C$ at each point.

☆New☆ Can use it to calculate directional derivatives.



$$\frac{\partial F}{\partial V}(p) = D_{V_p} F = V_p(F)$$

$$= F'(p) \cdot V = \nabla F(p) \cdot V$$

(instantaneous) = rate of change of F as you move away from

p with velocity v.

Examples

① $F(x,y,z) = x^2y - 2ze^{x-3z}$

ⓐ Find $\frac{\partial F}{\partial v}$ at $(3,2,1)$ in direction $v=(-2,1,1)$.

$$\nabla F = \left(2xy - 2ze^{x-3z},\ x^2 - 0,\ 0 - 2e^{x-3z} - 2z(-3)e^{x-3z}\right)$$

$$= \left(2xy - 2ze^{x-3z},\ x^2,\ -2e^{x-3z} + 6ze^{x-3z}\right)$$

$$\overset{(3,2,1)}{=} \left(2\cdot3\cdot2 - 2\cdot1\cdot1,\ 9,\ -2\cdot1 + 6\cdot1\cdot1\right)$$

$$= (10, 9, 4)$$

$$\frac{\partial F}{\partial v} = \nabla F \cdot v = (10,9,4)\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}$$

$$= -20 + 9 + 4 = \boxed{-7}.$$

(so the F is decreasing in that direction.)

(b) $F(x,y,z) = x^2 y - 2ze^{x-3z}$

Find $\frac{\partial F}{\partial v}$ at $(1,-1,2)$

in direction $(0,1,0)$.

From before:

$$\nabla F = \left(2xy - 2ze^{x-3z},\ x^2,\ -2e^{x-3z} + 6ze^{x-3z}\right)$$

$(1,-1,2)$

$$= \left(2(1)(-1) - 2(2)e^{1-6},\ 1,\ -2e^{-5} + 6(2)e^{-5}\right)$$

$$= \left(-2-4e^{-5},\ 1,\ -2e^{-5}+12e^{-5}\right)$$

$$\frac{\partial F}{\partial v} = \nabla F \cdot v$$

$$= \left(-2-4e^{-5},\ 1,\ -2e^{-5}+12e^{-5}\right)\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= \boxed{1}.$$

In this case $(0,1,0)$ is the unit vector in direction of increasing y $\Rightarrow \frac{\partial F}{\partial v} = \frac{\partial F}{\partial y} = x^2 \Big|_{(1,-1,2)} = 1$.

(c) Same Function, same point $(1,-1,2)$.
Find two examples of vectors v
where $\frac{\partial F}{\partial v}(1,-1,2) = 0$.

$\nabla F(1,-1,2) =$

$$(-2-4e^{-5},\ 1,\ -2e^{-5}+12e^{-5}).$$

$\nabla F \cdot v = 0$

eg: $v = (+1,\ 2+4e^{-5},\ 0)$

$v = (0,\ \underbrace{2e^{-5}-12e^{-5}},\ 1)$

<u>Comment</u>: These two vectors are $\perp$ to ∇F at $(1,-1,2)$ and thus are in the tangent plane to the level set $F(x,y,z) = \underbrace{F(1,-1,2)}_{c}$

<u>Comment</u>: Often people will only consider directional derivatives using unit vectors.

<u>Note</u>: to change a vector v into a unit vector, just divide by its norm.

eg. $v = (3,-4) \in \mathbb{R}^2$.

$$\begin{pmatrix}\text{unit vector in} \\ v \text{ direction}\end{pmatrix} = \frac{1}{\|(3,-4)\|}(3,-4)$$

$$= \frac{1}{\sqrt{9+16}}(3,-4) = \frac{1}{5}(3,-4)$$
$$= \left(\frac{3}{5}, -\frac{4}{5}\right).$$

<u>Example</u>: Consider the function $f(x,y) = 3x^2 + 3y^2$ in $\mathbb{R}^2$.
Find all directional derivatives of unit vectors at the point $(1,1)$.

Every unit vector is (v_1, v_2) with
$\sqrt{v_1^2+v_2^2} = 1 \Rightarrow v_1^2 + v_2^2 = 1$

ie $(v_1, v_2) = (\cos\theta, \sin\theta)$

$0 \le \theta \le 2\pi$

cover all possible unit vectors.

Our function is $f(x,y) = 3x^2 + 3y^2$

$$\nabla f = (6x, 6y) \overset{(1,1)}{=} (6,6).$$

$$\frac{\partial f}{\partial v} = \nabla f \cdot v = (6,6) \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$v = (\cos\theta, \sin\theta)$

$$= 6\cos\theta + 6\sin\theta$$

Questions: (a) Where is $\frac{\partial f}{\partial v}$ maximum?

(b) Where do $\frac{\partial f}{\partial v}$ minimum?

(c) Where is $\frac{\partial f}{\partial v}$ 0?

"Where" – in what directions? What θ?

$$\frac{\partial f}{\partial v} = 6\cos\theta + 6\sin\theta$$

. . . . . to be continued . . .